Hitchin Systems in Supersymmetric Field Theory III

Andrew Neitzke
Notes by Qiaochu Yuan

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Last time we saw that the Coulomb branch of $X_{\mathfrak{g}}[C_0 \times S^1_R]$ (where C_0 refers to taking the zero-area limit) is the Hitchin integrable system M and that the Coulomb branch of $X_{\mathfrak{g}}[C_0]$ itself is the base B of the Hitchin integrable system. We also saw that over nice points, the low energy physics on B was governed by abelian $N = 2$ SYM. The fibers of the Hitchin fibration are tori here. Over bad points we might get something more singular.

But why is there a map $M \to B$ at all? Let R be the ring of chiral local operators in $X_{\mathfrak{g}}[C_0]$. Taking vacuum expectation values / 1-point correlation functions of a local operator O gives a function $F_O : B \to \mathbb{C}$, and B itself turns out to be the Spec of this ring. But any local operator in the 4-dimensional theory gives a local operator in the 3-dimensional theory with the same operator product expansions, so we get a map $M \to \operatorname{Spec} R$, or equivalently a map $M \to B$.

Q: is it clear that a map that arises in this way should be surjective or have any other nice properties?

A: it's possible to construct a section of it in this case, namely the Hitchin zero section. In general supersymmetry could get broken.

Local operators can only detect B. To detect all of M we need more operators, and we have them. Take a SUSY line operator $L(\zeta)$ from the 4d theory in $X_{\mathfrak{g}}[C_0]$. (The parameter $\zeta \in \mathbb{CP}^1$ tells us which supersymmetry is being preserved.) Wrap it around the circle. This gives a function

$$F_{L(\zeta)} : M \ni v \mapsto \langle L(\zeta) \rangle_v \in \mathbb{C}. \tag{1}$$

In particular, let P be a loop on C, let $\zeta \in \mathbb{C}^\times$, and let R be a representation of G. We get a line operator $L_{P,R}(\zeta)$ coming from wrapping one dimension of a surface around the loop P. The resulting function

$$F_{L_{P,R}(\zeta)} : M \ni (D, \varphi) \mapsto \mathbb{C} \tag{2}$$

is the holonomy function

$$D, \varphi \mapsto \operatorname{tr}_R \operatorname{hol}_P \nabla(\zeta) \tag{3}$$

where

$$\nabla(\zeta) = \frac{\varphi}{\zeta} + D + \varphi^\dagger \zeta. \tag{4}$$

M has a $\mathbb{CP}^1$ worth of complex structure, and this is a holomorphic function on M with the complex structure corresponding to ζ. (The original complex structure corresponds to $\zeta = 0$.) If $\mathfrak{g} = A_1$ these functions give a linear basis for the algebra of holomorphic functions on M. It is sometimes called the skein algebra, and the line operators themselves should provide some kind of categorification of the skein algebra.

Q: what happens if $\Sigma = T^2$? We're supposed to get $N = 4$ SYM; do we?

A: in that case, every Higgs bundle is reducible, so the Higgs branch is sticking out of every point of the Coulomb branch. There are some positivity phenomena in the skein algebra which also seem to fail in this case. Apparently if you do this in the correct derived way then everything is still fine.

How do we actually compute $F_{L_{P,R(\zeta)}}(v)$? Say, given v on the real / zero section of M, labeled by a point $u \in B$. The following is joint work with Davide Gaiotto and Greg Moore.

The idea is to look for a low-energy description. We know that the low-energy physics of $X_{\mathfrak{g}}[C_0]$ is abelian gauge theory, so the low-energy description must also be a line operator in abelian gauge theory. In an abelian gauge theory we can classify all of the line defects, and there is one for every possible electromagnetic charges. For electric charges they're SUSY Wilson lines, and for magnetic charges they're SUSY 't Hooft lines (but it's enough to understand the Wilson lines). We had a lattice Γ_u of electromagnetic charges and we get one line defect for every element $\gamma \in \Gamma_u$. The corresponding operator is

$$\langle L_\gamma(\zeta) \rangle = \exp\left(\frac{RZ_\gamma}{\zeta} + i\theta_\gamma + R\overline{Z}_\gamma \zeta \right) \tag{5}$$

where the middle term is the holonomy of the abelian gauge fields and $Z_\gamma = \oint_\gamma \lambda$ where λ is the Liouville 1-form on Σ_u pulled back from $T^*(C)$.

In the nonabelian case, the claim is now that in $X_{\mathfrak{g}}[C_0]$, for any generic $(u, \zeta) \in B \times \mathbb{C}^\times$, there are a collection of integers

$$I(P, R, \gamma, u, \zeta) \in \mathbb{Z} \tag{6}$$

such that at low enough energies

$$L_{P,R}(\zeta) \sim \sum_{\gamma \in \Gamma} I(P, R, \gamma, u, \zeta) L_\gamma(\zeta). \tag{7}$$

We should think of these integers as the graded dimension / SUSY index of the charge-γ part of the BPS Hilbert space of $X_{\mathfrak{g}}[C]$ with a defect $L_{P,R}(\zeta)$ inserted. The above statement is telling us how to build an object in the category of line operators out of simpler objects.

This description breaks down at a real codimension-1 locus

$$\ell_\mu = \left\{ (u, \zeta) \mid \frac{Z_\mu(u)}{\zeta} \in \mathbb{R} \right\}. \tag{8}$$

What goes wrong is that some particle becomes massless. We can ask a more refined question with a line defect inserted, but then the locus at which things break down becomes much more complicated. It leads to a complicated network on the Hitchin base. The integers $I(P, R, \gamma, u, \zeta)$ are only piecewise constant and jump at the walls of the network.

After compactifying on S^1, we get

$$F_{L_{P,R}(\zeta)} = \sum_{\gamma\in\Gamma} I(P,R,\gamma,u,\zeta)\chi_\gamma(\zeta). \tag{9}$$

These functions $\chi_\gamma(\zeta)$ should satisfy

1. $\chi_\gamma\chi_\mu = \chi_{\gamma+\mu}(-1)^{\langle\gamma,\mu\rangle}$
2. $\chi_\gamma(\zeta) \sim \chi_\gamma^{sf}(\zeta)$ as $R\to\infty$
3. $\chi_\gamma(\zeta) \sim \chi_\gamma^{sf}(\zeta)$ as $\zeta\to 0,\infty$, where

$$\chi_\gamma^{sf}(\zeta) = \exp\left(\frac{RZ_\gamma}{\zeta} + R\overline{Z}_\gamma\zeta\right) \tag{10}$$

4. As (ζ,u) crosses a wall ℓ_μ, we have

$$\chi_\gamma(\zeta) \mapsto \chi_\gamma(\zeta)\prod_{n\geq 1}(1-\chi_{n\mu}(\zeta))^{n\ell(n\mu,u)} \tag{11}$$

   where $\ell(n\mu,u)$ is something we can compute.

(Various questions and answers happen here.)